Mid-Semester Exam   Representation Theory of groups
Maximum Time: 2:30   Maximum Marks: 35
Answer any five and each question is worth 7 marks

1. Let G be a topological group. Then prove that

   (a) G is discrete if and only if $\{e\}$ is open in G,

   (b) G is Hausdorff if and only if $\{e\}$ is closed in G.

2. (a) Let G be a topological group and F be a compact subset of G. Then for each open neighbourhood V of e there exists open neighbourhood W of e such that $xWx^{-1} \subset V$ for all $x \in F$.

   (b) Show that a compact group G has a basis of open invariant neighbourhoods at e, that is there exists a basis of open neighbourhoods $\{U_i\}$ at e such that $xU_ix^{-1} = U_i$ for all $x \in G$.

3. Suppose G is a topological group and the topology of G is given by a metric 'd' such that $d(ax, ay) = d(x, y)$ for all $a, x, y \in G$. Then

   (a) if G is locally compact, then the space (G, d) is complete;

   (b) if H is a subgroup of G and H is locally compact, then H is closed in G.

4. Let G be a locally compact Hausdorff group and Δ be the modular function. Then

   (a) show that Δ is a continuous homomorphism;

   (b) if $G = \overline{[G, G]}$, then prove that G is unimodular.

5. Let G be a locally compact Hausdorff group and μ be a left Haar measure on G.

   (a) Show that $\mu(E^{-1}) = \int_E \frac{1}{\Delta(x)} d\mu(x)$ for all Borel subsets E of G.

   (b) If G has a relatively compact open set K in G such that $xKx^{-1} = K$ for all $x \in G$, then show that $\mu(E^{-1}) = \mu(E)$ for all Borel sets E in G.

6. Let G be a finite group and $\rho: G \to GL(V)$ be a representation of G on a finite-dimensional vector space V over $\mathbb{C}$. Then

   (a) show that $\frac{1}{|G|} \sum_{g \in G} \rho(g)$ is a projection onto the space $V^G = \{v \in V \mid \rho(g)v = v\}$ and

   (b) show that $\sum_{g \in G} \chi_V(g)$ is a multiple of order of G.

7. Let $G = S_n$ be the symmetric group on n-letters. Let $T\colon G \to GL(n, \mathbb{C})$ be the map defined by $T(\sigma)(z_1, \cdots, z_n) = (z_{\sigma^{-1}1}, \cdots, z_{\sigma^{-1}n})$ for all $\sigma \in G$ and $z_i \in \mathbb{C}$. Then show that T is a representation of G and T is not irreducible if $n > 1$. For $n > 1$, show that T restricted to the invariant subspace $\{(z_1, \cdots, z_n) \mid \sum z_i = 0\}$ is irreducible.